Theorem: For a polynomial $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ there is no finite formula using only a, b, c, d, e, all complex numbers, addition, multiplication, division, and taking n'th roots, that always returns a root of that polynomial.

The proof will make sense if I do some examples first by proving weaker results so you can get the idea.

Proposition 1: There is no formula for a quadratic (degree 2 polynomial) equation using only addition, subtraction, multiplication and division of the coefficients.

Proof: The main proof builds on the idea we are about to use. Suppose there were, then we would have a continuous function from the coefficients to the roots that is single valued and therefore picks a root.

Now suppose this hypothetical formula lets us find the root 1 of $x^2 - 1 = 0$, or the root -1 as it would be the same idea. Now write the equation above as $x^2 = 1$. Now we know that if we have $x^2 = e^{i\theta}$ the roots are $x = \pm e^{\frac{i\theta}{2}}$. Now lets look at how the root 1 changes as we vary θ continuously from 0 to 2π. What would end up happening is the following:

$x^2 = e^{i0}$	Roots: $1, -1$
$x^2 = e^{i0.1}$	Roots: $e^{i0.05}, -e^{i0.05}$
…	…
$x^2 = e^{i2\pi}$	Roots: $e^{i\pi} = -1, -e^{i\pi} = 1$

The problem is, now, that the root returned by the formula must be the one on the left in the table above – it cannot change at any point in the process to being the one on the right because otherwise it would not be continuous. Therefore our formula would return both 1 and -1 for $x^2 = 1$ but our formula is single valued. Contradiction.

Definition 1: Suppose we have a function of one variable involving addition, multiplication and division and roots. Then a **bad point** of this function is a point such that any of the roots inside it or denominators inside it are 0.

Note that by standard power series properties, if we are not at a bad point we are adding, multiplying, and composing functions that are either roots or fractions, each of which has a local taylor series around non-zero inputs, and by level 6 power series properties the composition of functions with a local taylor series has a local taylor series.

Definition 2: Suppose we have a function of one variable involving addition, multiplication and division and roots. Then its **nesting depth** is how many layers of "root inside a root inside a root..." are allowed. For example, $\frac{x^2+2}{x^7-3x+3}$ has nesting depth 0, $x^4 + \sqrt{\frac{1+x^3}{x^5}}$ has nesting depth 1, and a function like $2x^6 + \sqrt[3]{\pi^2 x^4 + \sqrt[4]{x} - \sqrt{\frac{1+x^3}{x^5}}}$ has nesting depth 2 since it has roots inside roots but not roots inside roots inside roots. Nesting depth may depend on how we write the function but this will not matter for our purposes.

Lemma: Suppose we have a function of one variable involving addition, multiplication and division and roots that contains at least one non-bad point and is not always zero and has finite nesting depth. Then the "characteristic set" of the function, ie the union of its zeroes and its bad points are isolated, meaning that for all such points in the union, there is a circle around that point such that no other points are in the union, and that for all points in C not in the union, there is a circle around it with no points in the union.

Why this is useful: In the quadratic argument above if we could cross 0 we could switch to the negative square root and still be continuous so it would not work. We need to show that there is a path that leads to the contradiction without crossing 0.

Proof of lemma: We will do this by induction on the nesting depth.

For nesting depth 0, the most complicated expression we can get is one polynomial divided by another polynomial. The characteristic set of this function roots of the numerator or the denominator polynomial. There are finitely many of these, and finite sets of points are clearly isolated – The set of distances from one point to all points in the set that are not that point has a minimum since it is the minimum of finitely many finite things so with a ball of a smaller radius than that you are good.

Now suppose this theorem holds for all nesting depths up to k. Then a nesting depth k+1 expression looks like (in the most complicated case) $\frac{f(x)+\sqrt[i_1]{p_1}+\sqrt[i_2]{p_2}+\cdots+\sqrt[i_n]{p_n}}{g(x)+\sqrt[j_1]{q_1}+\sqrt[j_2]{q_2}+\cdots+\sqrt[j_m]{q_m}}$ where f and g are functions with nesting depth 0 and p_i, q_i are expressions of nesting depth k and i_r, j_r are integers greater than 1. Now any point in the characteristic set of this function is of one of the following types:

- In the characteristic set of one of the p's or q's (automatically a bad point)
- In the characteristic set of f or g (automatically a bad point)
- A zero of the denominator (automatically a bad point)
- A zero of the numerator (not necessarily a bad point but still in the characteristic set)

We know that the first two are isolated sets by the induction hypothesis. Now suppose we are considering the numerator or the denominator and it is at a point that is not a bad point of the numerator or denominator, then it has a local taylor series. If this point is a zero point, then write it as a taylor series: $c_k(x-x_0)^k + c_{k+1}(x-x_0)^{k+1} + \cdots = (x-x_0)^k(c_k + c_{k+1}(x-x_0) + \cdots)$, where c_k is the first non-zero term (I prove shortly a non-zero term always exists). Then $c_k + c_{k+1}(x-x_0) + \cdots$ is not 0 at x_0 and continuous there and thus not 0 locally around x_0. If we are not at a 0 point then apply the argument above with k is 0. Now I just have to prove that we don't have an all-zero taylor series about that point.

Now suppose the taylor series is all zero at a point, then since the taylor series is valid in an interval around that point the function is zero in an interval around that point. Let Z be the set of points with an all zero taylor series about it – assume this is non-empty for a contradiction. This taylor series is known to be valid on some interval – it converges everywhere and I have not claimed it is valid everywhere (although I will show this) but merely only valid on some circle with radius R. Then Z is open, ie has the property that for every point in Z there is a circle of points around it in Z, for example with a circle of radius $\frac{R}{2}$. Now consider U to be the set of C without the bad points, then U is open by the induction hypothesis (At any point a circle around it avoids a bad point). Also, if U\Z is non-empty then U\Z is open – For any point with a non-zero taylor series since the first non-zero term is continuous at that point it is non-zero in a ball around itself. Therefore U is a disjoint union of two open sets. Therefore U is a disjoint union of two open sets in $\mathbb{C}$. Now build a line segment from any point A in Z to any point B in U\Z. By the fact that $\mathbb{C}$\U is isolated this line segment contains finitely many points in U. Since the points are U are isolated we can dodge them by putting small semicircles in our path to get a continuous path A to B. Now consider the path as a continuous function F that ranges from 0 to A to 1 at B, for example by length travelled divided by total length. Now for any point C on the path in U, it must have be that there is a disc around that point in U, and thus an open interval around t must be a part of the path that is contained in A since the path is continuous so it does not jump out of that disc. But then at the point x in the path where $F(x) = Sup(F(t), t \in A)$, x cannot be in A otherwise it would be in A at a larger point, so x is in B, but then x is also in B in an interval around it, so we do not have the least upper bound of the points in A anymore. Contradiction, therefore one of Z and U\Z is empty, and if Z=U then the characteristic set is all of ℂ which contradicts the hypothesis of the lemma.

Therefore to prove the lemma all we need now is to prove that a finite union of sets of isolated points in ℂ is isolated. Luckily this is easy. Suppose we have a union of N sets of isolated points. Then at each point, for each of the N sets, that point is either one of the points that set and thus has a disc around it with no points, or simply has an interval around it with no points without being a point in that set. Either way, it has the kind of disc we want, so the intersection of all N of those discs is still a disc as N is finite and it is exactly the kind of disc we want.

Now we have this lemma. So suppose we have a function G that takes in the coefficients and outputs a number and its rule only involves +, -, *, /, $\sqrt[n]{x}$, and we consider the obvious function F from the roots to the coefficients (just by expanding the polynomial in terms of its roots). Now if we keep all but one root constant, F is a function of a single variable using just +, -, *, /, $\sqrt[n]{x}$, so by the lemma it has isolated bad points. This means we can move a root around in such a way that we dodge a bad point.

So here is what happens now. When we suppose we have a quintic formula we will suppose that it works at all its non-bad points, as if we can disprove that we are done. We make the same assumption for propositions we do about lower degree polynomials.

So now suppose we are at a non-bad point, then we can swap any two roots continuously in such a way that we do not cross a bad point and the roots do not cross. Since bad points are isolated, we can wiggle our two roots around in balls (their "safe balls") and still be at a non-bad point. Also, if we start with a bad point, we can get a root from any point to any other while keeping the other ones constant by simply going in a straight line and using a semicircle to dodge bad points or other roots (since we don't want roots to cross eachother, ok as there are finitely many of them). So to swap roots 1 and 2, move root 1 to somewhere that is not a bad point when keeping roots 2, 3, etc constant inside the safe ball of root 2, move root 2 to a non-bad point in the safe ball of where root 1 used to be (all balls have non-bad points by the isolation property). Now we can move root 2 to where root 1 was and root 1 to where root 2 was, and we never a bad point as now only root 1 is moved as everything is in exactly one safe ball – and the order of the roots does not matter as the coefficients remain unchanged.

We now prove a proposition about cubics and quartics to make it make sense what we will do with the quintic as it would not make sense if we straight up did it. Note that if in any of our hypothetical formulae that we suppose exist in future proofs, if the stuff we are saying is non-zero were always zero the formula would either be invalid or that part would be redundant and we could just remove it, so I'm not proving "oh a formula exists but that stuff must be always zero".

Proposition 2: Any formula for a cubic ($x^3 + ax^2 + bx + c = 0$) using only +, *, /, the complex numbers, and n'th roots must have nesting depth at least 2.

A formula with nesting depth 2 exists but it is very complicated (image below) so generally numerical methods are more practical.

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)+\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}}$$

$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)-\sqrt{\left(\frac{-b^3}{27a^3}+\frac{bc}{6a^2}-\frac{d}{2a}\right)^2+\left(\frac{c}{3a}-\frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}.$$

Image of the cubic formula for $ax^3 + bx^2 + cx + d = 0$, where which root we get depends on which cube root we use, noting that changing which square root we use does not change the result but merely the order of the terms. To prove this works, you can plug the formula into the polynomial and expand everything and check that it is 0. If you want to actually do this, then good luck. Obviously I'm not going to do it.

Proof (note that this concept is a bit difficult to understand): Suppose we had a formula with nesting depth 1 – since 0 can be ruled out by the quadratic argument. Then pick a root in the formula and call the stuff inside it X. X is continuous and single valued, since it just uses the basic math operations, and it is continuous along any path we use to permute the roots because we spent five million years ensuring that. Now take the roots and label them r_1, r_2 and r_3 and suppose that at this particular point our hypothetical formula outputs r_1. Now swap r_1 and r_2 continuously such that X avoids 0 and infinity. When we do this, X will go back to where it started – X is a function symmetric in the roots and we merely reordered the roots. If X is inside an n'th root, and wrapped around the origin m times, then $\sqrt[n]{X}$ will move an angle $2\pi\frac{m}{n}$ anticlockwise. This is why we needed to make sure we do not touch the origin – we either wrap around it some times or we do not, and we do not blow up to infinity. Now swap r_2 and r_3. This will cause X to possibly or possibly not wrap around the origin. Lets say it wraps p times anticlockwise, then $\sqrt[n]{X}$ will have moved a total angle of $2\pi\frac{m+p}{n}$ anticlockwise. Now permute the roots in cycle notation by doing the permutation (2 3)(1 2)(2 3)(1 2). This is the cycle (1 2 3) so it leaves the roots different, but $\sqrt[n]{X}$will have moved a total angle of $2\pi\frac{m+p-p-m}{n} = 0$ anticlockwise. This is a problem because all of the roots in our hypothetical equation, and thus the value of the equation, will be the same. We supposed it returned r_1, it never switched to returning another root since the roots moved continuously and never crossed and the function from the roots to the coefficients to X is continuous, and the roots never crossed to allow it to jump. So our formula returns r_1, which moved, but also returns the same thing that it returned at the start since it ended up back where it started. Contradiction.

Proposition 3: Any formula for a quartic ($x^4 + ax^3 + bx^2 + cx + d = 0$) using only +, *, /, the complex numbers, and n'th roots must have nesting depth at least 2.

For interest, there is a formula as follows:

$$x = -\frac{a}{4} \pm_1 \frac{1}{2}\sqrt{\frac{a^2}{4}-\frac{2b}{3}+\frac{\sqrt[3]{2b^3-9abc-72db+27c^2+27a^2d+\sqrt{(2b^3-9abc-72db+27c^2+27a^2d)^2-4(b^2-3ac+12d)^3}}}{3\sqrt[3]{2}}+\frac{\sqrt[3]{2}(b^2-3ac+12d)}{3\sqrt[3]{2b^3-9abc-72db+27c^2+27a^2d+\sqrt{(2b^3-9abc-72db+27c^2+27a^2d)^2-4(b^2-3ac+12d)^3}}}} \pm_2 \frac{1}{2}\sqrt{\frac{a^2}{2}-\frac{4b}{3}-\frac{\sqrt[3]{2b^3-9abc-72db+27c^2+27a^2d+\sqrt{(2b^3-9abc-72db+27c^2+27a^2d)^2-4(b^2-3ac+12d)^3}}}{3\sqrt[3]{2}}-\frac{\sqrt[3]{2}(b^2-3ac+12d)}{3\sqrt[3]{2b^3-9abc-72db+27c^2+27a^2d+\sqrt{(2b^3-9abc-72db+27c^2+27a^2d)^2-4(b^2-3ac+12d)^3}}} \mp_1 \frac{-a^3+4ab-8c}{4\sqrt{\frac{a^2}{4}-\frac{2b}{3}+\frac{\sqrt[3]{2b^3-9abc-72db+27c^2+27a^2d+\sqrt{(2b^3-9abc-72db+27c^2+27a^2d)^2-4(b^2-3ac+12d)^3}}}{3\sqrt[3]{2}}+\frac{\sqrt[3]{2}(b^2-3ac+12d)}{3\sqrt[3]{2b^3-9abc-72db+27c^2+27a^2d+\sqrt{(2b^3-9abc-72db+27c^2+27a^2d)^2-4(b^2-3ac+12d)^3}}}}}}$$

Where $\pm_1$ and $\mp_1$ have opposite signs and $\pm_2$ is independent of them Two independent instances of plusminus gives us four possibilities like we want. Obviously, numerical methods are more practical. This is the exact form for the larger root of $x^4 - x - 1 = 0$:

$$x = \frac{1}{2}\sqrt{\frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{849}\right)}}{3^{2/3}} - 4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}} + \frac{1}{2}\sqrt{4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}} - \frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{849}\right)}}{3^{2/3}} + \frac{2}{\sqrt{\frac{\sqrt[3]{\frac{1}{2}\left(9+\sqrt{849}\right)}}{3^{2/3}} - 4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\right)}}}}}$$

Image of the exact form from wolframalpha

This actually has nesting depth 4 but we will prove at least 3 is needed.

Proof:

The idea we did for proposition 2 (Permutation A, B, A backwards then B backwards) is called a commutator. This is commonly used on the Rubik's cube. We will exploit this idea more.

So we can do a commutator to rule out nesting depth 1 like we did for the cubic. This leaves any roots without roots inside them unchanged, but we do not know what it does to everything inside the root with a nesting depth 1 expression inside it – we just know that after a commutator the stuff inside the root is back to where it started, but it could have wrapped around the origin in the process. But now lets do a commutator of commutators and track the nesting depth 1 expressions. We know a commutator of swaps is a 3-cycle. A commutator of three cycles: $(1\ 2\ 3)(2\ 3\ 4)(1\ 3\ 2)(2\ 4\ 3) = (1\ 4)(2\ 3)$. This will make, for the same reason, the number of total times the nesting depth 1 expressions wrap around the origin be 0, therefore any nesting depth 2 expression will be back where it started and we have the same contradiction.

Proof of the main theorem:

This is easy once you have understood the concepts used above. We just need to rule out any finite nesting depth. But we can have an infinite chain of commutators of commutators of commutators of … This is because a certain commutator of three cycles has the property $(1\ 2\ 3)(1\ 4\ 5)(1\ 3\ 2)(1\ 5\ 4) = (1\ 2\ 4)$, which is another 3-cycle. Therefore we can make a chain as long as we want where at each step we just have a 3-cycle, and we can pick this in a way that it moves the root our hypothetical formula outputs. We can therefore get the same contradiction. This completes the proof.

Addendum: Even if we include sin, cos and exp, there is still no formula, as the assumption was merely continuity, and these functions have taylor series that work everywhere so the assumptions of the lemma still work.